Asymptotic results for random flights

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Abstract

The random flights are (continuous time) random walks with finite velocity. Often, these models describe the stochastic motions arising in biology. In this paper we study the large time asymptotic behavior of random flights. We prove the large deviation principle for conditional laws given the number of the changes of direction, and for the non-conditional laws of some standard random flights.

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1 Introduction

The random flight models have been introduced in order to describe the real motions and they can be connected with the biological locomotion. The original formulation of the random flight problem is due to Pearson, who considers a random walk with fixed and constant steps. The Pearson's model deals with a random walker moving in the plane in straight lines with fixed length and turning through any angle whatever. This random walk has been introduced by Pearson for modeling the migration of mosquitoes invading cleared jungles. Over the years the random flights have attracted the attention of different researchers and the original model has been extended randomizing the displacements of the walk; see for example [10], [8], [4], [9], [1].

Several experiments have highlighted that many aspects of the cell biology are governed by stochastic motions. For instance, some motile bacteria, such as *Escherichia coli* and *Salmonella typhimurium*, and blood cells have motion which is well-approximated by random straight-lines with discrete changes of direction (see [6], and references therein). Therefore, the random flights seem to be suitable for the analytic characterization of the behavior of these micro-organisms.

In this work, we focus our attention on two random flights in $\mathbb{R}^d$ indicated by $\{\underline{X}_d(t) : t \geq 0\}$, with $d \geq 2$, and $\{\underline{Y}_d(t) : t \geq 0\}$, with $d \geq 3$, respectively. Now we briefly describe these models; for more details the reader can consult [1]. The two random motions start at the origin, move with finite velocity c and, when n changes of direction are recorded, the $n+1$ time lengths of the displacements performed by the motions have suitable rescaled Dirichlet distributions. More precisely, let us indicate by τ_j, with $j \in \{1, ..., n+1\}$ and $\tau_0 = 0$, $\tau_{n+1} = t - \sum_{j=1}^{n} \tau_j$, the time displacements of the motions. We assume that

$$f_X(\tau_1, \dots, \tau_n) = \frac{\Gamma((n+1)(d-1))}{(\Gamma(d-1))^{n+1}} \frac{1}{t^{(n+1)(d-1)-1}} \prod_{j=1}^{n+1} \tau_j^{d-2}$$

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for $\{\underline{X}_d(t) : t \geq 0\}$ (the rescaled Dirichlet distributions with parameters $(d-1, \ldots, d-1)$; see eq. (1.4) in [1]) and

$$f_Y(\tau_1, \ldots, \tau_n) = \frac{\Gamma((n+1)(\frac{d}{2}-1))}{(\Gamma(\frac{d}{2}-1))^{n+1}} \frac{1}{t^{(n+1)(\frac{d}{2}-1)-1}} \prod_{j=1}^{n+1} \tau_j^{\frac{d}{2}-2}$$

for $\{\underline{Y}_d(t) : t \geq 0\}$ (the rescaled Dirichlet distributions with parameters $\left(\frac{d}{2}-1, \ldots, \frac{d}{2}-1\right)$; see eq. (1.5) in [1]) where, in both cases, $0 < \tau_j < t - \sum_{k=0}^{j-1} \tau_k$ for $j \in \{1, \ldots, n\}$. Moreover, in both cases, the orientations concerning the displacements - the initial one, and the ones after each change of direction - are uniformly chosen on the sphere of $\mathbb{R}^d$ with radius one (see eq. (1.3) in [1]). Any direction is chosen independently from the previous one. The sample paths of $\{\underline{X}_d(t) : t \geq 0\}$, and $\{\underline{Y}_d(t) : t \geq 0\}$ appear as random straight-lines with sharp turns. Figure 1 displays a typical sample path.

The aim of this work is to investigate the asymptotic behavior of the above models as $t \to \infty$. In particular we obtain large deviation results concerning the random flights. The large deviation theory provides an asymptotic computation of small probabilities on exponential scale; estimates based on large deviations play a crucial role in resolving a variety of questions in statistics, engineering, statistical mechanics and applied probability.

The paper is organized as follows. In Section 2, we provide a brief introduction on the large deviation theory and recall the probability distributions for the random flights considered in this work. The main results are contained in Section 3. In particular we obtain the large deviation results for the random flights $\{\underline{X}_d(t) : t \geq 0\}$, with $d \geq 2$, and $\{\underline{Y}_d(t) : t \geq 0\}$, with $d \geq 3$. If we consider a homogeneous Poisson process governing the changes of direction of the motion, we obtain a standard random flight. We have that $\{\underline{X}_2(t) : t \geq 0\}$ and $\{\underline{Y}_4(t) : t \geq 0\}$ represent the standard cases. Section 3 contains some propositions concerning the standard random flights. In this last case we are also able to consider the large deviations results for non-conditional laws. Many comparisons concerning rate functions of the standard random flights are listed in Section 4.

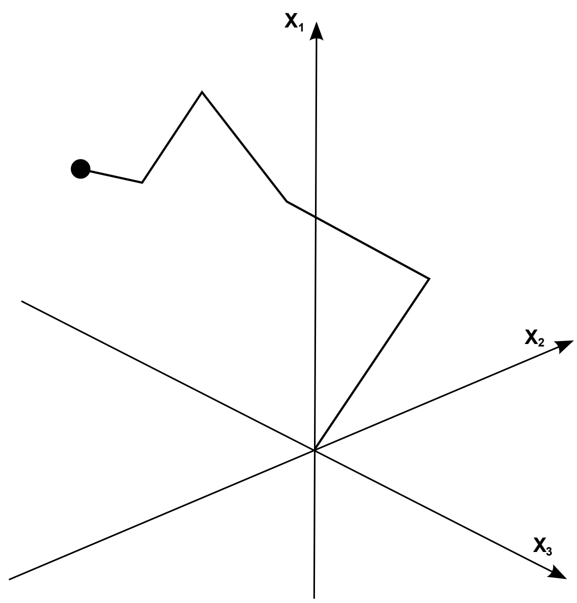

Figure 1: A sample path in $\mathbb{R}^3$ consisting of $n = 4$ changes of direction.

2 Preliminaries

We start with some preliminaries on large deviations. We recall the basic definitions in [3] (pages 4–5). Let $\mathcal{Z}$ be a Hausdorff topological space with Borel σ-algebra $\mathcal{B}_{\mathcal{Z}}$. A lower semi-continuous function $I : \mathcal{Z} \to [0, \infty]$ is called rate function. A family of probability measures $\{\pi_t : t > 0\}$ on $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$ satisfies the *large deviation principle* (LDP for short), as $t \to \infty$, with rate function I if

$$\limsup_{t\to\infty} \frac{1}{t} \log \pi_t(F) \leq -\inf_{z\in F} I(z) \quad \text{for all closed sets } F$$

and

$$\liminf_{t\to\infty} \frac{1}{t} \log \pi_t(G) \geq -\inf_{z\in G} I(z) \quad \text{for all open sets } G.$$

A rate function I is said to be good if all the level sets $\{\{z \in \mathcal{Z} : I(z) \leq \gamma\} : \gamma \geq 0\}$ are compact. In what follows we use condition (b) with equation (1.2.8) in [3], which is equivalent to the lower bound for open sets:

$$\liminf_{t\to\infty} \frac{1}{t} \log \pi_t(G) \geq -I(z) \qquad \begin{array}{l}\text{for all } z \in \mathcal{Z} \text{ such that } I(z) < \infty \text{ and}\\ \text{for all open sets } G \text{ such that } z \in G.\end{array} \tag{1}$$

Remark 2.1. *Assume that $\{\pi_t^{(1)} : t > 0\}$ and $\{\pi_t^{(2)} : t > 0\}$ satisfy the LDP with the rate functions I_1 and I_2, respectively; moreover assume that I_1 and I_2 uniquely vanish at the same point z_0. Then, if we have $I_1(z) > I_2(z)$ for all z in a neighborhood U of z_0 (except z_0 because $I_1(z_0) = I_2(z_0) = 0$), we can say that $\{\pi_t^{(1)} : t > 0\}$ converges to z_0 faster than $\{\pi_t^{(2)} : t > 0\}$, as $t \to \infty$. Indeed, for all $\varepsilon > 0$, there exists t_ε such that $\frac{\pi_t^{(1)}(U^c)}{\pi_t^{(2)}(U^c)} \leq e^{-t(I_1(U^c)-I_2(U^c)+2\varepsilon)}$ for all $t > t_\varepsilon$, where $I_k(U^c) = \inf_{z\in U^c} I_k(z)$ for $k \in \{1, 2\}$; thus, since $I_1(U^c) > I_2(U^c) > 0$, we have $\frac{\pi_t^{(1)}(U^c)}{\pi_t^{(2)}(U^c)} \to 0$ as $t \to \infty$.*

Throughout the paper we use the following notation: $\|\underline{z}_d\| := \sqrt{z_1^2 + \cdots + z_d^2}$ is the norm of $\underline{z}_d = (z_1, \ldots, z_d) \in \mathbb{R}^d$; $B_\delta(\underline{z}_d) := \{\underline{y}_d \in \mathbb{R}^d : \|\underline{y}_d - \underline{z}_d\| < \delta\}$ is the neighborhood of $\underline{z}_d$ with radius $\delta > 0$; $\underline{0}_d$ is the null vector in $\mathbb{R}^d$.

Now we recall some preliminaries on the densities obtained in Theorem 2 in [1]. They concern two random flights in $\mathbb{R}^d$: $\{\underline{X}_d(t) : t \geq 0\}$ for $d \geq 2$; $\{\underline{Y}_d(t) : t \geq 0\}$ for $d \geq 3$. Moreover the number $n \geq 1$ of changes of direction in the time interval $[0, t]$ is fixed and, from now on, we use the following notation:

$$\mu_d(E, t; n) := P_n(\underline{X}_d(t) \in E); \quad \nu_d(E, t; n) := P_n(\underline{Y}_d(t) \in E).$$

Then we have:

$$\mu_d(E, t; n) := \int_{E\cap B_{ct}(\underline{0}_d)} \frac{\Gamma(\frac{n+1}{2}(d-1) + \frac{1}{2})}{\Gamma(\frac{n}{2}(d-1))} \frac{(c^2t^2 - \|\underline{x}_d\|^2)^{\frac{n}{2}(d-1)-1}}{\pi^{d/2}(ct)^{(n+1)(d-1)-1}} dx_1 \cdots dx_d \quad (d \geq 2);$$

$$\nu_d(E, t; n) := \int_{E\cap B_{ct}(\underline{0}_d)} \frac{\Gamma((n+1)(\frac{d}{2}-1) + 1)}{\Gamma(n(\frac{d}{2}-1))} \frac{(c^2t^2 - \|\underline{y}_d\|^2)^{n(\frac{d}{2}-1)-1}}{\pi^{d/2}(ct)^{2(n+1)(\frac{d}{2}-1)}} dy_1 \cdots dy_d \quad (d \geq 3).$$

Note that in both cases we have a probability measure $\xi_d(\cdot, t; n)$ defined by

$$\xi_d(E, t; n) = \int_{E\cap B_{ct}(\underline{0}_d)} h_d(\underline{z}_d, t; n) dz_1 \cdots dz_d \tag{2}$$

for some density $h_d(\cdot, t; n)$ which has the universal isotropic form

$$h_d(\underline{z}_d, t; n) = \alpha(n) t^{-\gamma(n)} (c^2t^2 - \|\underline{z}_d\|^2)^{\beta(n)} \text{ (for } \underline{z}_d \in B_{ct}(\underline{0}_d)), \tag{3}$$

more precisely we have

$$\begin{cases} \underline{X}_d \ \rightsquigarrow \ \gamma(n) = (n+1)(d-1) - 1, & \alpha(n) = \frac{\Gamma(\frac{n+1}{2}(d-1)+\frac{1}{2})}{\Gamma(\frac{n}{2}(d-1))\pi^{d/2}c^{\gamma(n)}}, \quad \beta(n) = \frac{n}{2}(d-1) - 1; \\ \underline{Y}_d \ \rightsquigarrow \ \gamma(n) = 2(n+1)(\frac{d}{2}-1), & \alpha(n) = \frac{\Gamma((n+1)(\frac{d}{2}-1)+1)}{\Gamma(n(\frac{d}{2}-1))\pi^{d/2}c^{\gamma(n)}}, \quad \beta(n) = n(\frac{d}{2}-1) - 1. \end{cases} \tag{4}$$

A standard random flight $\{Z_d(t) : t \geq 0\}$ in $\mathbb{R}^d$ is a random motion which starts at the origin $\underline{0}_d$, moves with constant velocity c, chooses the directions uniformly on the d-dimensional sphere with radius 1, and changes direction at any occurrence of a homogeneous Poisson process $\{N(t) : t \geq 0\}$ with intensity λ. This last assumption implies that the joint distribution of the lengths between two consecutive changes of direction is uniform. A reference for this random motion is [8]; the case $d = 2$ was studied in some earlier papers (see e.g. section 2 in [10] where $c = 1$).

One can check that some laws presented above are suitable conditional distributions for standard random flights; more precisely, for each fixed $t > 0$, we have

$$\mu_2(\cdot, t; n) = P(Z_2(t) \in \cdot | N(t) = n) \quad \text{and} \quad \nu_4(\cdot, t; n) = P(Z_4(t) \in \cdot | N(t) = n).$$

Indeed, for $d = 2$ and $d = 4$ the Dirichlet distributions reduce to the uniform distribution.

Finally, in view of what follows, it is useful to recall the following limits (which can be proved by inspection).

Lemma 2.2. *If we have* $\lim_{t\to\infty} w_t = w$ *for some* $w \in (0, \infty)$*, then*

$$b(w) := \lim_{t\to\infty} \frac{\beta(tw_t)}{t} = \begin{cases} \frac{w}{2}(d-1) & if\ \xi_d(\cdot, t; n) = \mu_d(\cdot, t; n) \\ w(\frac{d}{2}-1) & if\ \xi_d(\cdot, t; n) = \nu_d(\cdot, t; n) \end{cases}$$

and $\lim_{t\to\infty} \frac{\gamma(tw_t)}{t} = 2b(w)$.

3 Results

We start with Proposition 3.1 which provides the LDP for the families of laws $\{\mu_d(t\cdot, t; n_t) : t > 0\}$ and $\{\nu_d(t\cdot, t; n_t) : t > 0\}$, as $t \to \infty$, under a suitable limit condition on n_t.

Proposition 3.1. *Assume to have* $\lim_{t\to\infty} w_t = w$ *for some* $w \in (0, \infty)$*. Then:*
(a) $\{\mu_d(t\cdot, t; tw_t) : t > 0\}$ *satisfies the LDP with good rate function* $I_d(\cdot; w)$ *defined by*

$$I_d(\underline{z}_d; w) = \begin{cases} w(d-1) \log\left(\frac{c}{\sqrt{c^2 - \|\underline{z}_d\|^2}}\right) & if\ \|\underline{z}_d\| < c \\ \infty & otherwise; \end{cases}$$

(b) $\{\nu_d(t\cdot, t; tw_t) : t > 0\}$ *satisfies the LDP with good rate function* $J_d(\cdot; w)$ *defined by*

$$J_d(\underline{z}_d; w) = \begin{cases} 2w\left(\frac{d}{2}-1\right) \log\left(\frac{c}{\sqrt{c^2 - \|\underline{z}_d\|^2}}\right) & if\ \|\underline{z}_d\| < c \\ \infty & otherwise. \end{cases}$$

Proof. We consider $\{\xi_d(t\cdot, t; n_t) : t > 0\}$ defined by (2)-(3), and we take into account (4) for each case. Thus the rate functions $I_d(\cdot; w)$ and $J_d(\cdot; w)$ in the statement coincide with

$$K_d(\underline{z}_d; w) = \begin{cases} 2b(w)\log\left(\frac{c}{\sqrt{c^2-\|\underline{z}_d\|^2}}\right) & \text{if } \|\underline{z}_d\| < c \\ \infty & \text{otherwise,} \end{cases}$$

where $b(w)$ is as in Lemma 2.2. We remark that, by Lemma 2.2, we have $\lim_{t\to\infty}\frac{1}{t}\log\alpha(tw_t) = -2b(w)\log c$. The proof is divided in two parts.

1) Proof of the lower bound for open sets. We want to check the equivalent condition (1), i.e.

$$\liminf_{t\to\infty}\frac{1}{t}\log\xi_d(tG, t; tw_t) \geq -K_d(\underline{z}_d; w)$$

for all $\underline{z}_d \in \mathbb{R}^d$ such that $\|\underline{z}_d\| < c$ and for all open sets G such that $\underline{z}_d \in G$. Firstly we can take $\varepsilon > 0$ small enough to have $B_\varepsilon(\underline{z}_d) \subset G \cap B_c(\underline{0}_d)$. Then we have

$$\begin{aligned}\xi_d(tG, t; tw_t) \geq&\xi_d(tB_\varepsilon(\underline{z}_d), t; tw_t) = \xi_d(B_{\varepsilon t}(\underline{z}_d t), t; tw_t)\\ =&\int_{B_{\varepsilon t}(\underline{z}_d t)}\alpha(tw_t)t^{-\gamma(tw_t)}(c^2t^2 - \|\underline{v}_d\|^2)^{\beta(tw_t)}dv_1\cdots dv_d\\ =&\int_{B_\varepsilon(\underline{z}_d)}\alpha(tw_t)t^{-\gamma(tw_t)}(c^2t^2 - \|\underline{v}_d\|^2t^2)^{\beta(tw_t)}t^d dv_1\cdots dv_d\\ \geq&\alpha(tw_t)t^{-\gamma(tw_t)+2\beta(tw_t)+d}(c^2 - \sup\{\|\underline{v}_d\|^2 : \underline{v}_d \in B_\varepsilon(\underline{z}_d)\})^{\beta(tw_t)}\text{measure}(B_\varepsilon(\underline{z}_d)),\end{aligned}$$

whence we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\log\xi_d(tG, t; tw_t) \geq -2b(w)\log c + b(w)\log(c^2 - \sup\{\|\underline{v}_d\|^2 : \underline{v}_d \in B_\varepsilon(\underline{z}_d)\}),$$

and we conclude by letting ε go to zero.

2) Proof of the upper bound for closed sets. We have to check

$$\limsup_{t\to\infty}\frac{1}{t}\log\xi_d(tF, t; tw_t) \leq -\inf_{\underline{z}_d\in F}K_d(\underline{z}_d; w) \quad \text{for all closed sets } F.$$

Firstly note that this condition trivially holds if $\underline{0}_d \in F$ and if $F \cap B_c(\underline{0}_d) = \emptyset$. Thus, from now on, we assume that $\underline{0}_d \notin F$ and $F \cap B_c(\underline{0}_d) \neq \emptyset$. We can find $\underline{z}_d^F \in F$ such that $\|\underline{z}_d^F\| = \inf\{\|\underline{z}_d\| : \underline{z}_d \in F \cap B_c(\underline{0}_d)\}$; note that $r_F := \|\underline{z}_d^F\| \in (0, c)$. Then, since $F \subset (B_{r_F}(\underline{0}_d))^c$, we have

$$\begin{aligned}\xi_d(tF, t; tw_t) \leq \xi_d((B_{r_Ft}(\underline{0}_d))^c, t; tw_t) &= \int_{r_Ft}^{ct}\alpha(tw_t)t^{-\gamma(tw_t)}(c^2t^2 - \rho^2)^{\beta(tw_t)}\frac{2\pi^{d/2}}{\Gamma(d/2)}\rho^{d-1}d\rho\\ &\leq \alpha(tw_t)t^{-\gamma(tw_t)}\frac{\pi^{d/2}}{\Gamma(d/2)}(ct)^{d-2}\int_{r_Ft}^{ct}(c^2t^2 - \rho^2)^{\beta(tw_t)}2\rho d\rho\\ &= \alpha(tw_t)t^{-\gamma(tw_t)}\frac{\pi^{d/2}}{\Gamma(d/2)}(ct)^{d-2}\left[-\frac{(c^2t^2 - \rho^2)^{\beta(tw_t)+1}}{\beta(tw_t)+1}\right]_{\rho=r_Ft}^{\rho=ct}\\ &= \alpha(tw_t)t^{-\gamma(tw_t)+2\beta(tw_t)+2}\frac{\pi^{d/2}}{\Gamma(d/2)}(ct)^{d-2}\frac{(c^2 - r_F^2)^{\beta(tw_t)+1}}{\beta(tw_t)+1}\end{aligned}$$

whence we obtain

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log\xi_d(tF, t; tw_t) &\leq -2b(w)\log c + b(w)\log(c^2 - r_F^2) = -2b(w)\log\left(\frac{c}{\sqrt{c^2 - r_F^2}}\right)\\ &= -2b(w)\log\left(\frac{c}{\sqrt{c^2 - \|\underline{z}_d^F\|^2}}\right) = -K_d(\underline{z}_d^F; w).\end{aligned}$$

In conclusion, since $K_d(\underline{z}_d; w) \geq$ (resp. $=$)$K_d(\underline{u}_d; w)$ if and only if $\|\underline{z}_d\| \geq$ (resp. $=$)$\|\underline{u}_d\|$, we have $K_d(\underline{z}_d^F; w) = \inf_{\underline{z}_d \in F} K_d(\underline{z}_d; w)$ and this completes the proof. $\square$

The next Propositions 3.2-3.3 provide the LDP for the non-conditional laws $\{P(Z_2(t) \in \cdot) : t > 0\}$ and $\{P(Z_4(t) \in \cdot) : t > 0\}$ and, in analogy with the symbols introduced above, from now on we use the following notation:

$$\mu_2(\cdot, t) = P(Z_2(t) \in \cdot); \quad \nu_4(\cdot, t) = P(Z_4(t) \in \cdot).$$

We start with the first family of laws: for each fixed $t > 0$, it is known (see e.g. eq. (1.2) in [8]) that $\mu_2(\cdot, t)$ has an absolutely continuous part given by

$$\frac{\lambda}{2\pi c} \frac{e^{-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - \|\underline{z}_2\|^2}}}{\sqrt{c^2t^2 - \|\underline{z}_2\|^2}} 1_{B_{ct}(\underline{0}_2)}(\underline{z}_2) dz_1 dz_2, \tag{5}$$

and a singular part uniformly distributed on the boundary of $B_{ct}(\underline{0}_2)$ with weight $e^{-\lambda t}$.

Proposition 3.2. *The family $\{\mu_2(t\cdot, t) : t > 0\}$ satisfies the LDP with good rate function I_2 defined by*

$$I_2(\underline{z}_2) = \begin{cases} \lambda\left(1 - \sqrt{1 - \frac{\|\underline{z}_2\|^2}{c^2}}\right) & if\ \|\underline{z}_2\| \leq c \\ \infty & otherwise. \end{cases}$$

Proof. The proof is divided in two parts.
1) Proof of the lower bound for open sets. We want to check the equivalent condition (1), i.e.

$$\liminf_{t\to\infty} \frac{1}{t} \log \mu_2(tG, t) \geq -\lambda\left(1 - \sqrt{1 - \frac{\|\underline{z}_2\|^2}{c^2}}\right)$$

for all $\underline{z}_2 \in \mathbb{R}^2$ such that $\|\underline{z}_2\| \leq c$ and for all open sets G such that $\underline{z}_2 \in G$. Firstly we can take $\varepsilon > 0$ small enough to have $B_\varepsilon(\underline{z}_2) \subset G$; moreover, if $\underline{z}_2 \in B_c(\underline{0}_2)$, we also require that $B_\varepsilon(\underline{z}_2) \subset B_c(\underline{0}_2)$. Then we have

$$\begin{aligned}
\mu_2(tG, t) \geq& \mu_2(tB_\varepsilon(\underline{z}_2), t) = \mu_2(B_{\varepsilon t}(\underline{z}_2 t), t) \\
\geq& \int_{B_{\varepsilon t}(\underline{z}_2 t)\cap B_{ct}(\underline{0}_2)} \frac{\lambda}{2\pi c} \frac{e^{-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - \|\underline{v}_2\|^2}}}{\sqrt{c^2t^2 - \|\underline{v}_2\|^2}} dv_1 dv_2 \\
\geq& \int_{B_\varepsilon(\underline{z}_2)\cap B_c(\underline{0}_2)} \frac{\lambda}{2\pi c} \frac{e^{-\lambda t + \frac{\lambda t}{c}\sqrt{c^2 - \|\underline{v}_2\|^2}}}{ct} t^2 dv_1 dv_2 \\
\geq& \frac{\lambda t}{2\pi c} \frac{e^{-\lambda t + \lambda t\sqrt{1 - \frac{\sup\{\|\underline{v}_2\|^2 : \underline{v}_2 \in B_\varepsilon(\underline{z}_2)\cap B_c(\underline{0}_2)\}}{c^2}}}}{c} \cdot \underbrace{\mathrm{measure}(B_\varepsilon(\underline{z}_2) \cap B_c(\underline{0}_2))}_{>0},
\end{aligned}$$

and, arguing as in the proof of Proposition 3.1, we conclude by taking $\liminf_{t\to\infty} \frac{1}{t}\log$ (for both the left hand side and the right hand side) and by letting ε go to zero.
2) Proof of the upper bound for closed sets. We have to check

$$\limsup_{t\to\infty} \frac{1}{t} \log \mu_2(tF, t) \leq -\inf_{\underline{z}_2 \in F} I_2(\underline{z}_2) \quad \text{for all closed sets } F.$$

Firstly note that this condition trivially holds if $\underline{0}_2 \in F$ and if $F \cap \overline{B_c(\underline{0}_2)} = \emptyset$. Thus, from now on, we assume that $\underline{0}_2 \notin F$ and $F \cap \overline{B_c(\underline{0}_2)} \neq \emptyset$. We can find $\underline{z}_2^F \in F$ such that $\|\underline{z}_2^F\| = \inf\{\|\underline{z}_2\| : \underline{z}_2 \in F \cap \overline{B_c(\underline{0}_2)}\}$; note that $r_F := \|\underline{z}_2^F\| \in (0, c]$. Then, since $F \subset (B_{r_F}(\underline{0}_2))^c$, we have

$$\mu_2(tF,t) \leq \mu_2((B_{r_F t}(\underline{0}_2))^c, t) = \int_{r_F t}^{ct} \frac{\lambda}{2\pi c} \frac{e^{-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2-\rho^2}}}{\sqrt{c^2t^2-\rho^2}} 2\pi\rho d\rho + e^{-\lambda t}$$
$$= \left[-e^{-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2-\rho^2}}\right]_{\rho=r_F t}^{\rho=ct} + e^{-\lambda t} = \exp\left(-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - r_F^2t^2}\right),$$

whence we obtain

$$\limsup_{t\to\infty} \frac{1}{t} \log \mu_2(tF,t) \leq -\lambda + \frac{\lambda}{c}\sqrt{c^2 - r_F^2} = -\lambda\left(1 - \sqrt{1 - \frac{\|\underline{z}_2^F\|^2}{c^2}}\right) = -I_2(\underline{z}_2^F).$$

Thus, arguing as in the proof of Proposition 3.1, we conclude noting that $I_2(\underline{z}_2^F) = \inf_{\underline{z}_2 \in F} I_2(\underline{z}_2)$ because $I_2(\underline{z}_2) \geq$ (resp. $=$)$I_2(\underline{u}_2)$ if and only if $\|\underline{z}_2\| \geq$ (resp. $=$)$\|\underline{u}_2\|$. □

Now we consider the second family of laws: for each fixed $t > 0$, it is known (see e.g. Theorem 3.2 in [8]) that $\nu_4(\cdot, t)$ has an absolutely continuous part given by

$$\frac{\lambda}{c^4t^3\pi^2} e^{-\frac{\lambda}{c^2t}\|\underline{z}_4\|^2} \left(2 + \frac{\lambda}{c^2t}\left(c^2t^2 - \|\underline{z}_4\|^2\right)\right) 1_{B_{ct}(\underline{0}_4)}(\underline{z}_4) dz_1dz_2dz_3dz_4, \tag{6}$$

and a singular part uniformly distributed on the boundary of $B_{ct}(\underline{0}_4)$ with weight $e^{-\lambda t}$.

Proposition 3.3. *The family $\{\nu_4(t\cdot, t) : t > 0\}$ satisfies the LDP with good rate function J_4 defined by*

$$J_4(\underline{z}_4) = \begin{cases} \frac{\lambda}{c^2}\|\underline{z}_4\|^2 & \textit{if } \|\underline{z}_4\| \leq c \\ \infty & \textit{otherwise.} \end{cases}$$

Proof. The proof is divided in two parts.
1) Proof of the lower bound for open sets. We want to check the equivalent condition (1), i.e.

$$\liminf_{t\to\infty} \frac{1}{t} \log \nu_4(tG,t) \geq -\frac{\lambda}{c^2}\|\underline{z}_4\|^2$$

for all $\underline{z}_4 \in \mathbb{R}^4$ such that $\|\underline{z}_4\| \leq c$ and for all open sets G such that $\underline{z}_4 \in G$. Firstly we can take $\varepsilon > 0$ small enough to have $B_\varepsilon(\underline{z}_4) \subset G$; moreover, if $\underline{z}_4 \in B_c(\underline{0}_4)$, we also require that $B_\varepsilon(\underline{z}_4) \subset B_c(\underline{0}_4)$. Then we have

$$\begin{aligned}\nu_4(tG,t) \geq& \nu_4(tB_\varepsilon(\underline{z}_4), t) = \nu_4(B_{\varepsilon t}(\underline{z}_4 t), t)\\
\geq& \int_{B_{\varepsilon t}(\underline{z}_4 t)\cap B_{ct}(\underline{0}_4)} \frac{\lambda}{c^4t^3\pi^2} e^{-\frac{\lambda}{c^2t}\|\underline{v}_4\|^2}\left(2 + \frac{\lambda}{c^2t}\left(c^2t^2 - \|\underline{v}_4\|^2\right)\right) dv_1dv_2dv_3dv_4\\
=& \int_{B_\varepsilon(\underline{z}_4)\cap B_c(\underline{0}_4)} \frac{\lambda}{c^4t^3\pi^2} e^{-\frac{\lambda t}{c^2}\|\underline{v}_4\|^2}\left(2 + \frac{\lambda t}{c^2}\left(c^2 - \|\underline{v}_4\|^2\right)\right) t^4 dv_1dv_2dv_3dv_4\\
\geq& \frac{\lambda t}{c^4\pi^2} e^{-\frac{\lambda t}{c^2}\sup\{\|\underline{v}_4\|^2 : \underline{v}_4 \in B_\varepsilon(\underline{z}_4)\cap B_c(\underline{0}_4)\}}\\
&\cdot\left(2 + \frac{\lambda t}{c^2}\left(c^2 - \sup\{\|\underline{v}_4\|^2 : \underline{v}_4 \in B_\varepsilon(\underline{z}_4) \cap B_c(\underline{0}_4)\}\right)\right) \underbrace{\mathrm{measure}(B_\varepsilon(\underline{z}_4) \cap B_c(\underline{0}_4))}_{>0},\end{aligned}$$

and we conclude following the lines of Proposition 3.2 (final part of the proof of the lower bound).
2) Proof of the upper bound for closed sets. We have to check

$$\limsup_{t\to\infty}\frac{1}{t}\log\nu_4(tF,t)\leq-\inf_{\underline{z}_4\in F}J_4(\underline{z}_4)\quad\text{for all closed sets }F.$$

Firstly note that this condition trivially holds if $\underline{0}_4\in F$ and if $F\cap\overline{B_c(\underline{0}_4)}=\emptyset$. Thus, from now on, we assume that $\underline{0}_4\notin F$ and $F\cap\overline{B_c(\underline{0}_4)}\neq\emptyset$. We can find $\underline{z}_4^F\in F$ such that $\|\underline{z}_4^F\|=\inf\{\|\underline{z}_4\|:\underline{z}_4\in F\cap\overline{B_c(\underline{0}_4)}\}$; note that $r_F:=\|\underline{z}_4^F\|\in(0,c]$. Then, since $F\subset(B_{r_F}(\underline{0}_4))^c$, we have

$$\nu_4(tF,t)\leq\nu_4((B_{r_Ft}(\underline{0}_4))^c,t)=\int_{r_Ft}^{ct}\frac{\lambda}{c^4t^3\pi^2}e^{-\frac{\lambda}{c^2t}\rho^2}\left(2+\frac{\lambda}{c^2t}(c^2t^2-\rho^2)\right)2\pi^2\rho^3d\rho+e^{-\lambda t};$$

moreover we note that, if $\rho\in[r_Ft,ct]$, we have

$$\begin{cases}0\leq\frac{\lambda}{c^2t}(c^2t^2-\rho^2)=\lambda t\left(1-\left(\frac{\rho}{ct}\right)^2\right)\leq\lambda t\\ 0\leq\frac{2\lambda\rho^3}{c^4t^3}=2\left(\frac{\rho}{ct}\right)^2\frac{\lambda\rho}{c^2t}\leq\frac{2\lambda\rho}{c^2t},\end{cases}$$

whence we obtain

$$\begin{aligned}\nu_4(tF,t)\leq&(2+\lambda t)\int_{r_Ft}^{ct}\frac{2\lambda\rho}{c^2t}e^{-\frac{\lambda}{c^2t}\rho^2}d\rho+e^{-\lambda t}\\ =&(2+\lambda t)\left[-e^{-\frac{\lambda}{c^2t}\rho^2}\right]_{\rho=r_Ft}^{\rho=ct}+e^{-\lambda t}=(2+\lambda t)e^{-\lambda\frac{r_F^2}{c^2}t}\left(1-e^{-\lambda t\left(1-\frac{r_F^2}{c^2}\right)}\right)+e^{-\lambda t};\end{aligned}$$

actually, if $r_F=c$, we have $\nu_4((B_{r_Ft}(\underline{0}_4))^c,t)=P(N(t)=0)=e^{-\lambda t}$. In conclusion, by Lemma 1.2.15 in [3] (we need this for $r_F\in(0,c)$; the case $r_F=c$ is trivial), we obtain

$$\limsup_{t\to\infty}\frac{1}{t}\log\nu_4(tF,t)\leq-\frac{\lambda}{c^2}r_F^2=-J_4(\underline{z}_4^F);$$

then we can conclude following the lines of Proposition 3.2 (final part of the proof of the upper bound). □

We remark that the rate function J_4 in the previous Proposition 3.3 is quadratic on $\overline{B_c(\underline{0}_4)}$; quadratic rate functions typically come up in the case of LDPs for Gaussian random variables and here, if we neglect the factor $2+\frac{\lambda}{c^2t}\left(c^2t^2-\|\underline{z}_4\|^2\right)$, the density in (6) is proportional to a Gaussian density restricted on $B_{ct}(\underline{0}_4)$.

4 Minor results and concluding remarks

We start with the comparison between the rate functions for the conditional laws and the non-conditional laws related to the standard random flights $\{Z_2(t):t\geq0\}$ and $\{Z_4(t):t\geq0\}$, in the spirit of Remark 2.1. In both cases the convergence of the conditional distributions is faster than the convergence of the non-conditional distributions if and only if $w\geq\lambda$. For the case $d=2$ we can refer to a result for the telegraph process on the real line, i.e. Proposition 2.3 in [2] (and its consequences), specified to the case $c_1=c_2=c$: more precisely we have to consider the rate functions $I_{\lambda,\lambda,c,c}^X$ (for the non-conditional laws) and the $I_{\lambda,\lambda,c,c}^{X|N}(\cdot;w)$ (for the conditional laws) in [2], and the equalities

$$\begin{cases}I_2(\underline{z}_2)=I_{\lambda,\lambda,c,c}^X(\|\underline{z}_2\|)\\ I_2(\underline{z}_2;w)=I_{\lambda,\lambda,c,c}^{X|N}(\|\underline{z}_2\|;w)\ (\text{for all }w>0)\end{cases}\quad\text{for all }\underline{z}_2\in\mathbb{R}^2.\tag{7}$$

For the case $d=4$ we have the following similar result.

Proposition 4.1. *We have two cases. (i) For $w \geq \lambda$, we have $J_4(\underline{z}_4) \leq J_4(\underline{z}_4; w)$ for all $\underline{z}_4 \in \mathbb{R}^4$; moreover the inequality is strict for $\underline{z}_4 \in \overline{B_c(\underline{0}_4)} \setminus \{\underline{0}_4\}$. (ii) For $w \in (0, \lambda)$, there exists $\gamma \in (\xi, 1)$ where $\xi = \sqrt{1 - \frac{w}{\lambda}}$ such that: $J_4(\underline{z}_4; w) > J_4(\underline{z}_4)$ for $\|\underline{z}_4\| \in (\gamma c, c]$, $J_4(\underline{z}_4; w) < J_4(\underline{z}_4)$ for $\|\underline{z}_4\| \in (0, \gamma c)$ and $J_4(\underline{z}_4; w) = J_4(\underline{z}_4)$ otherwise.*

Proposition 4.1 can be proved by inspection and we omit the details. Some inequalities in Proposition 4.1 between J_4 and $J_4(\cdot; w)$ are displayed in Figure 2 below when $\lambda = c = 1$.

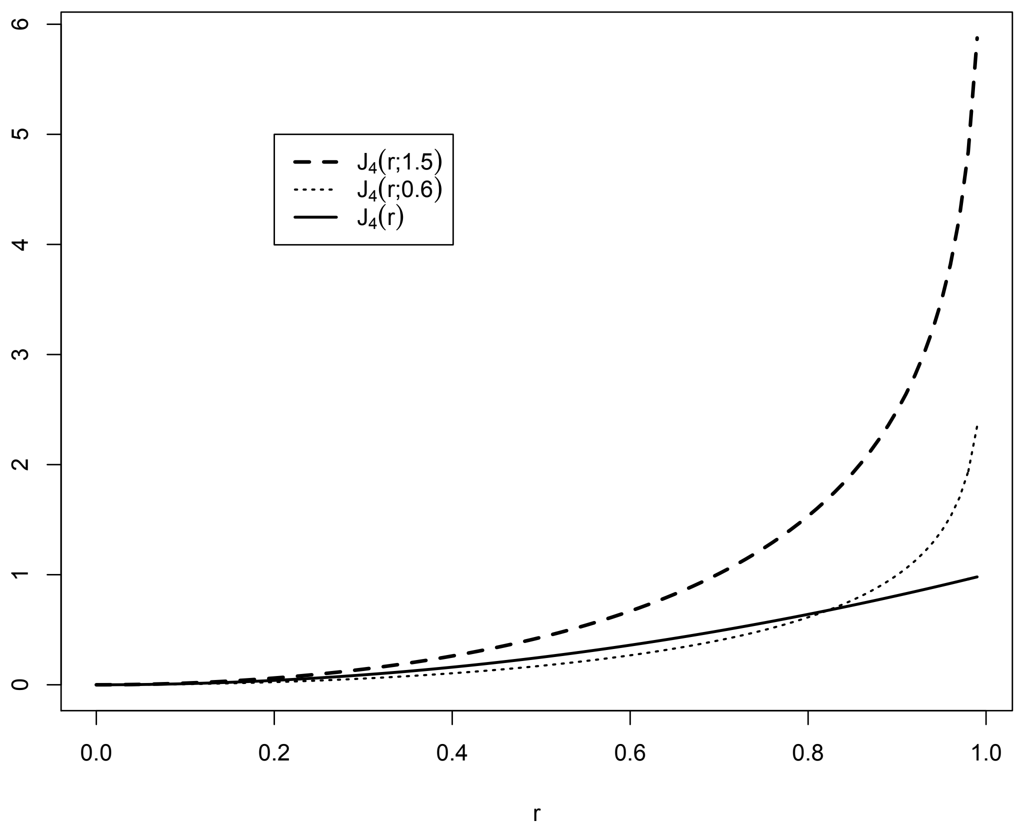

Figure 2: The functions $J_4(\underline{z}_4)$ and $J_4(\underline{z}_4; w)$ for $\lambda = c = 1$, where $r = \|\underline{z}_4\| \in [0, 1)$. Two choices for w: $w = 1.5 \geq 1$; $w = 0.6 \in (0, 1)$.

It is known (see the discussion in [8]) that the telegraph process without drift on the line has a strict analogy with the standard random flight in $\mathbb{R}^2$ and not in $\mathbb{R}^4$. Actually the density in (5) satisfies the two-dimensional telegraph equation (see e.g. the displayed equation just after (1.2) in [8]). We have a similar situation when we compare the rate functions for the telegraph process (without drift) on the line and for the standard random flights. Firstly we have (7) for the case $d = 2$. On the contrary, for $d = 4$, for all $\underline{z}_4 \in \mathbb{R}^4$ we have:

$$J_4(\underline{z}_4) = \begin{cases} \frac{\lambda}{c^2}\|\underline{z}_4\|^2 & \text{if } \|\underline{z}_4\| \leq c \\ \infty & \text{otherwise;} \end{cases} \quad I^X_{\lambda,\lambda,c,c}(\|\underline{z}_4\|) = \begin{cases} \lambda\left(1 - \sqrt{1 - \frac{\|\underline{z}_4\|^2}{c^2}}\right) & \text{if } \|\underline{z}_4\| \leq c \\ \infty & \text{otherwise;} \end{cases}$$

$$J_4(\underline{z}_4; w) = 2I^{X|N}_{\lambda,\lambda,c,c}(\|\underline{z}_4\|; w) \text{ for all } w > 0.$$

Therefore $J_4(\cdot)$ and $I^X_{\lambda,\lambda,c,c}(\|\cdot\|)$ are quite different, while $J_4(\cdot;w)$ and $I^{X|N}_{\lambda,\lambda,c,c}(\|\cdot\|;w)$ differ for the multiplicative factor 2.

In view of what follows we remark that the values assumed by the functions $I_2, I_2(\cdot;w), J_4, J_4(\cdot;w)$ depend on the distance of their arguments from the origin; therefore we use the symbol r in place of $\|\underline{z}_2\|$ and $\|\underline{z}_4\|$, and we write $I_2(r), I_2(r;w), J_4(r), J_4(r;w)$ instead of $I_2(\underline{z}_2), I_2(\underline{z}_2;w), J_4(\underline{z}_4), J_4(\underline{z}_4;w)$ with a slight abuse of notation. Then, for all $r \geq 0$, we have

$$\begin{cases} J_4(r;w) = 2I_2(r;w) \geq I_2(r;w) \\ J_4(r) \geq I_2(r) \end{cases}$$

(the first inequality is trivial, the second one can be checked with easy computations), and Figure 3 below displays the second inequality when $\lambda = c = 1$. We also remark that the two inequalities above turn into an inequality if and only if we have one the following cases:

$$J_4(r;w) = I_2(r;w) = \begin{cases} 0 & \text{if } r = 0 \\ \infty & \text{if } r \in [c,\infty); \end{cases} \qquad J_4(r) = I_2(r) = \begin{cases} 0 & \text{if } r = 0 \\ \lambda & \text{if } r = c \\ \infty & \text{if } r \in (c,\infty). \end{cases}$$

In particular we remark that $J_4(c) = I_2(c) = \lambda$ for the non-conditional laws concerns the case without changes of direction. In conclusion, by taking into account Remark 2.1, the convergence at the origin in $\mathbb{R}^4$ is faster than the analogous convergence at the origin in $\mathbb{R}^2$ (for both conditional and non-conditional laws); in some sense this is not surprising because one can expect to have a faster convergence of a normalized random flight in a higher space.

We point out another difference between the cases $d = 2$ and $d = 4$. We recall that, if we consider the telegraph process without drift, i.e. $\lambda_1 = \lambda_2 = \lambda$ and $c_1 = c_2 = c$, the telegraph equation converges to the heat equation as $\lambda \to \infty$ and $\frac{c^2}{\lambda} \to \sigma^2$ (this and other connections with the Brownian motion can be found in [7], section 4). The corresponding convergence of the large deviation rate functions under the same scaling is illustrated in [5] (subsection 4.2); more precisely the convergence of the large deviation rate functions proved there concerns the more general case with drift, and there are also convergence results for the decay rates of suitable level crossing probabilities. Here we remark that, if we consider the same limits for the rate functions in Propositions 3.2-3.3 (again as $\lambda \to \infty$ and $\frac{c^2}{\lambda} \to \sigma^2$), we have $I_2(\underline{z}_2) \to \frac{\|\underline{z}_2\|^2}{2\sigma^2}$ and $J_4(\underline{z}_4) \to \frac{\|\underline{z}_4\|^2}{\sigma^2}$; then, if we consider the rate function H_d defined by $H_d(\underline{z}_d) = \frac{\|\underline{z}_d\|^2}{2\sigma^2}$ for the LDP of $\left\{\frac{B_d(\sigma^2)}{\sqrt{t}} : t > 0\right\}$, where B_d is a standard d-dimensional centered Brownian motion, $H_2(\underline{z}_2)$ coincides with the limit for $I_2(\underline{z}_2)$, while $H_4(\underline{z}_4)$ is slightly different from the limit for $J_4(\underline{z}_4)$.

Finally, motivated by potential applications in biology, it is interesting to present the asymptotic lower bounds in (8) for the exit probabilities

$$\Psi_{Z_d}(t;r) = P(\{\|Z_d(s)\| > r \text{ for some } s \in [0,t]\}) \text{ for } d \in \{2,4\}.$$

Actually, for all $r \in (0,c)$, we have

$$\liminf_{t\to\infty} \frac{1}{t} \log \Psi_{Z_d}(t;rt) \geq -\ell(d), \text{ where } \ell(d) := \begin{cases} \lambda\left(1 - \sqrt{1 - \frac{r^2}{c^2}}\right) & \text{for } d = 2 \\ \lambda\frac{r^2}{c^2} & \text{for } d = 4 \end{cases} \tag{8}$$

by the inequality $\Psi_{Z_d}(t;rt) \geq P\left(\frac{\|Z_d(t)\|}{t} > r\right)$ (for all $t, r > 0$) and by the LDPs in Propositions 3.2-3.3.

For instance, r could represent a critical threshold in the analysis of the behavior of the bacteria. In some context is realistic to assume that the motile cells moving far from the starting point lose their intrinsic properties. The above lower bound permits us to provide some information about this event.

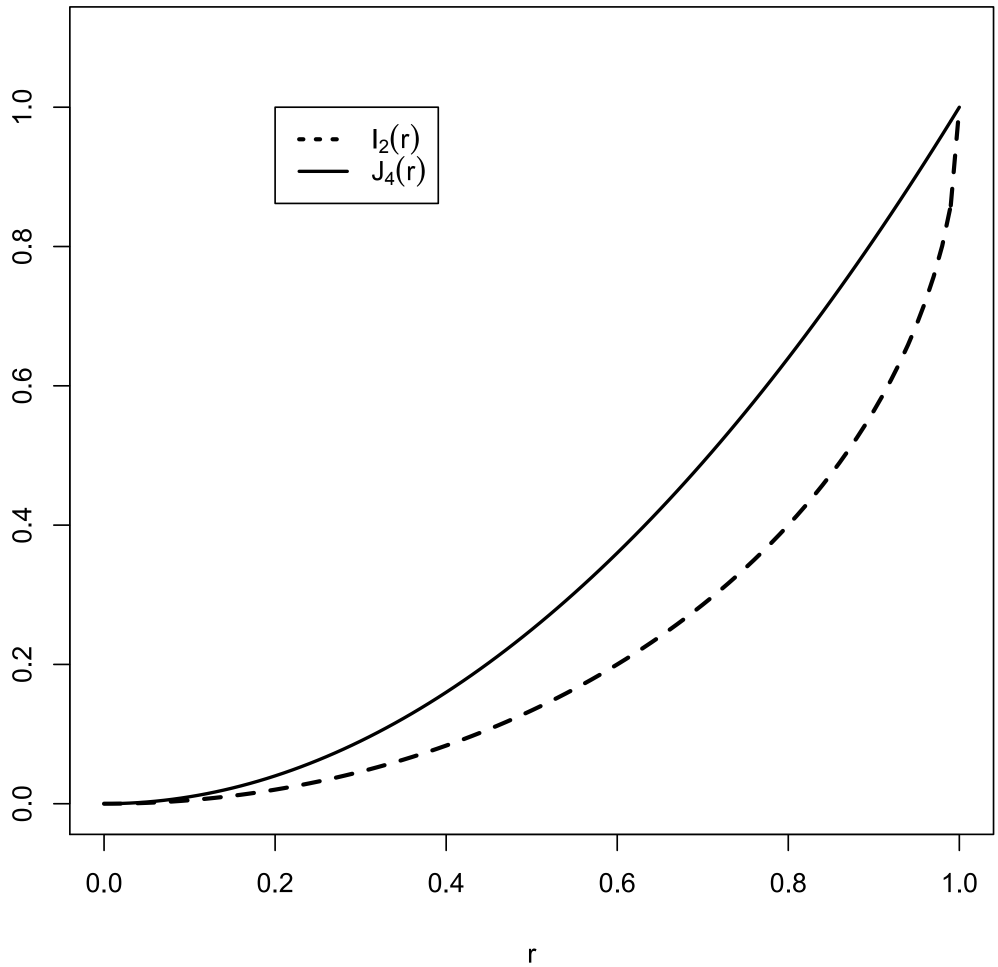

Figure 3: The functions $I_2(r)$ and $J_4(r)$ for $\lambda = c = 1$, where $r = \|\underline{z}_d\| \in [0, 1]$.

Obviously, if it is possible to observe the changes of direction of the random motion, then it would be good to consider the analogous asymptotic lower bounds by referring to the conditional laws (thus by considering Proposition 3.1 instead of Propositions 3.2-3.3).

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